

# LANDAU–GINZBURG HODGE NUMBERS FOR MIRRORS OF DEL PEZZO SURFACES

VALERY LUNTS AND VICTOR PRZYJALKOWSKI

ABSTRACT. We consider the conjectures from [KKP14] about Landau–Ginzburg Hodge numbers associated to tamely compactifiable Landau–Ginzburg models. We test these conjectures in case of dimension two, verifying some and giving a counterexample to the other.

## 1. INTRODUCTION

Homological Mirror Symmetry (HMS) conjecture of Kontsevich is a master conjecture which is expected to have many numerical consequences. Such numerical predictions can be formulated and tested in cases where HMS has not yet been established. In the paper [KKP14] the authors define three types of Hodge theoretical invariants

$$f^{p,q}(Y, w), \quad h^{p,q}(Y, w), \quad i^{p,q}(Y, w)$$

of tamely compactifiable (see Definitions 1 and 3) Landau–Ginzburg models  $w: Y \rightarrow \mathbb{C}$ . The numbers  $f^{p,q}(Y, w)$  come from the sheaf cohomology of certain logarithmic forms, the numbers  $h^{p,q}(Y, w)$  come from mirror considerations, and the numbers  $i^{p,q}(Y, w)$  come from ordinary mixed Hodge theory.

It is proved in [KKP14] that these numbers satisfy the identities

$$(1) \quad \dim H^m(Y, Y_b; \mathbb{C}) = \sum_{p+q=m} i^{p,q}(Y, w) = \sum_{p+q=m} h^{p,q}(Y, w) = \sum_{p+q=m} f^{p,q}(Y, w),$$

where  $Y_b$  is a smooth fiber of  $w$ . And the following refinements if (1) is conjectured based on HMS considerations:

$$(2) \quad f^{p,q}(Y, w) = h^{p,q}(Y, q) = i^{p,q}(Y, w).$$

Tamely compactifiable Landau–Ginzburg models  $(Y, w)$  typically appear as mirrors of projective Fano manifolds  $X$ . HMS predicts an equivalence of triangulated categories

$$D^b(\text{coh } X) \simeq FS(Y, w, \omega_Y),$$

where  $D^b(\text{coh } X)$  is a bounded derived category of coherent sheaves on  $X$  and  $FS(Y, w, \omega_Y)$  is a Fukaya–Seidel category of  $(Y, w)$  for an appropriately chosen symplectic form  $\omega_Y$ . In this situation the authors of [KKP14] make an additional conjecture

$$(3) \quad f^{p,q}(Y, w) = h^{p,n-q}(X),$$

where  $n = \dim X = \dim Y$ .

In this paper we test conjectures (2) and (3) in the case  $\dim Y = 2$ , i.e. when  $Y$  is a specific rational surface with a map  $w: Y \rightarrow \mathbb{C}$  such that the generic fiber is an elliptic curve. In this case we prove the equality  $f^{p,q}(Y, q) = h^{p,q}(Y, w)$  and give an example, where  $i^{p,q}(Y, w) \neq h^{p,q}(Y, w)$ . It is interesting to find a “correct” definition of numbers  $i^{p,q}(Y, w)$  which would be compatible with conjecture (2). Moreover, in case the Landau–Ginzburg is mirror to a del Pezzo surface  $X$  (see [AKO06]) we prove that  $f^{p,q}(Y, w) = h^{p,2-q}(X)$ , thus verifying conjecture (3).

Actually we first correct slightly the definition of the numbers  $h^{p,q}(Y, w)$  since the original definition in [KKP14] is clearly not what the authors had in mind. We hope that our methods can be used in testing conjectures (1) and (3) in higher dimensions.

The paper is organized as follows. In Section 2 we give the main definitions and formulate the conjectures that we consider following [KKP14]. In particular we recall definitions of the numbers  $f^{p,q}(Y, w)$ ,  $h^{p,q}(Y, w)$ , and  $i^{p,q}(Y, w)$ . We also formulate the main theorem of the paper (Theorem 11). In Section 3 we study monodromy of Landau–Ginzburg models. In the following sections we consider the case of dimension 2. In Section 4 we study topology and cohomological properties of the elliptic surfaces that we are interested in. In Section 5 we compute Landau–Ginzburg Hodge numbers for the elliptic surfaces and prove Proposition 24 and Proposition 30, which (together with Remark 31) give a proof of Theorem 11. A big part of this section is computations of  $f$ -adopted log forms that are needed for the numbers  $f^{p,q}(Y, w)$ . Finally in Section 6 we discuss our results. In particular we discuss a counterexample to a part of conjectures from [KKP14] related to numbers  $i^{p,q}(Y, w)$ .

**Notation and conventions.** All varieties are defined over the field of complex numbers  $\mathbb{C}$  and we consider them as topological spaces with the classical analytic topology. For a pair of topological spaces  $Y_0 \subset Y$  symbols like  $H^\bullet(Y)$ ,  $H^\bullet(Y, Y_0)$  will denote the singular cohomology (resp. relative singular cohomology) with coefficients in  $\mathbb{C}$ . Symbols like  $H_c^\bullet(Y)$  will denote the cohomology with compact supports (of  $Y$  with coefficients in the constant sheaf  $\mathbb{C}_Y$ ).

**Acknowledgments.** The authors are grateful to V. Golyshev, A. Kasprzyk, L. Katzarkov, and D. Orlov for useful discussions. Special thanks to T. Pantev and V. Turaev for their help. V. Lunts was partially supported by the NSA grant H98230-15-1-0255; V. Przyjalkowski has been funded by the Russian Academic Excellence Project “5-100” and was also partially supported by the grants RFFI 15-01-02158, RFFI 15-01-02164, RFFI 14-01-00160, RFFI 15-51-50045, MK-6019.2016.1.

## 2. COMPACTIFIED LANDAU–GINZBURG MODELS AND HODGE-THEORETICAL CONJECTURES

Let us recall some numerical conjectures from [KKP14] which are supposed to follow from the conjectural Homological Mirror Symmetry between Fano manifolds and Landau–Ginzburg models.

**Definition 1.** A *Landau–Ginzburg model* is a pair  $(Y, w)$ , where

- (1)  $Y$  is a smooth complex quasi-projective variety with trivial canonical bundle  $K_Y$ ;
- (2)  $w: Y \rightarrow \mathbb{A}^1$  is a morphism with a compact critical locus  $\text{crit}(w) \subset Y$ .

*Remark 2.* Note that there are no conditions on singularities of fibers.

Following [KKP14] we assume that there exists a *tame* compactification of the Landau–Ginzburg model as defined below.

**Definition 3.** A *tame compactified Landau–Ginzburg model* is the data  $((Z, f), D_Z)$ , where

- (1)  $Z$  is a smooth projective variety and  $f: Z \rightarrow \mathbb{P}^1$  is a flat morphism.
- (2)  $D_Z = (\cup_i D_i^h) \cup (\cup_j D_j^v)$  is a reduced normal crossings divisor such that
  - (i)  $D^v = \cup_j D_j^v$  is a scheme theoretical pole divisor of  $f$ , i.e.  $f^{-1}(\infty) = D^v$ . In particular  $\text{ord}_{D_j^v}(f) = -1$  for all  $j$ ;
  - (ii) each component  $D_i^h$  of  $D^h = \cup_i D_i^h$  is smooth and horizontal for  $f$ , i.e.  $f|_{D_i^h}$  is a flat morphism;
  - (iii) The critical locus  $\text{crit}(f) \subset Z$  does not intersect  $D^h$ .

(3)  $D_Z$  is an anticanonical divisor on  $Z$ .

One says that  $((Z, f), D_Z)$  is a *compactification of the Landau–Ginzburg model*  $(Y, w)$  if in addition the following holds:

(4)  $Y = Z \setminus D_Z$ ,  $f|_Y = w$ . We denote by  $j : Y \hookrightarrow Z$  the open embedding.

*Remark 4.* In [KKP14] one requires in above definitions an additional choice of compatible holomorphic volume forms on  $Z$  and  $Y$ . Since these forms will play no role in our work we omitted them.

Assume that we are given a Landau–Ginzburg model  $(Y, w)$  with a tame compactification  $((Z, f), D_Z)$  as above. We denote by  $n = \dim Y = \dim Z$  the (complex) dimension of  $Y$  and  $Z$ . Choose a point  $b \in \mathbb{A}^1$  which is near  $\infty$  and such that the fiber  $Y_b = w^{-1}(b) \subset Y$  is smooth. In [KKP14] the authors define geometrically three sets of what they call “Hodge numbers”  $i^{p,q}(Y, w)$ ,  $h^{p,q}(Y, w)$ ,  $f^{p,q}(Y, w)$ . Let us recall the definitions.

**2.1. The numbers  $f^{p,q}(Y, w)$ .** Recall the definition of the logarithmic de Rham complex  $\Omega_Z^\bullet(\log D_Z)$ . Namely,  $\Omega_Z^s(\log D_Z) = \wedge^s \Omega_Z^1(\log D_Z)$  and  $\Omega_Z^1(\log D_Z)$  is a locally free  $\mathcal{O}_Z$ -module generated locally by

$$\frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k}, dz_{k+1}, \dots, dz_n$$

if  $z_1 \cdots z_k = 0$  is a local equation of the divisor  $D_Z$ . Hence in particular  $\Omega_Z^0(\log D_Z) = \mathcal{O}_Z$ .

The numbers  $f^{p,q}(Y, w)$  are defined using the subcomplex  $\Omega_Z^\bullet(\log D_Z, f) \subset \Omega_Z^\bullet(\log D_Z)$  of *f-adapted forms*, which we recall next.

**Definition 5.** For each  $a \geq 0$  define a sheaf  $\Omega_Z^a(\log D_Z, f)$  of *f-adapted logarithmic forms* as a subsheaf of  $\Omega_Z^a(\log D_Z)$  consisting of forms which stay logarithmic after multiplication by  $df$ . Thus

$$\Omega_Z^a(\log D_Z, f) = \{\alpha \in \Omega_Z^a(\log D_Z) \mid df \wedge \alpha \in \Omega_Z^{a+1}(\log D_Z)\},$$

where one considers  $f$  as a meromorphic function on  $Z$  and  $df$  is viewed as a meromorphic 1-form.

**Definition 6.** The *Landau–Ginzburg Hodge numbers*  $f^{p,q}(Y, w)$  are defined as follows:

$$f^{p,q}(Y, w) = \dim H^p(Z, \Omega_Z^q(\log D_Z, f)).$$

**2.2. The numbers  $h^{p,q}(Y, w)$ .** Let  $N : V \rightarrow V$  be a nilpotent operator on a finite dimensional vector space  $V$  such that  $N^{m+1} = 0$ . Recall that this data defines a canonical (monodromy) *weight filtration centered at  $m$* ,  $W = W_\bullet(N, m)$  of  $V$

$$0 \subset W_0(N, w) \subset W_1(N, w) \subset \dots \subset W_{2m-1}(N, m) \subset W_{2m}(N, m) = V$$

with the properties

- (1)  $N(W_i) \subset W_{i-2}$ ,
- (2) the map  $N^l : gr_{m+l}^{W, m} V \rightarrow gr_{m-l}^{W, m} V$  is an isomorphism for all  $l \geq 0$ .

Let  $S^1 \simeq C \subset \mathbb{P}^1$  be a loop passing through the point  $b$  that goes once around  $\infty$  in the counter clockwise direction in such a way that there are no singular points of  $w$  on or inside  $C$ . It gives the monodromy transformation

$$T : H^\bullet(Y_b) \rightarrow H^\bullet(Y_b)$$

and also the corresponding monodromy transformation on the relative cohomology

$$(4) \quad T : H^\bullet(Y, Y_b) \rightarrow H^\bullet(Y, Y_b).$$

in such a way that the sequence

$$\dots \rightarrow H^m(Y, Y_b) \rightarrow H^m(Y) \rightarrow H^m(Y_b) \rightarrow H^{m+1}(Y, Y_b) \rightarrow \dots$$

is  $T$ -equivariant, where  $T$  acts trivially on  $H^\bullet(Y)$ . (See Section 3 for the construction and the discussion of the monodromy transformation  $T: H^\bullet(Y, Y_b) \rightarrow H^\bullet(Y, Y_b)$ ). Since we assume that the infinite fiber  $f^{-1}(\infty) \subset Z$  is a reduced divisor with normal crossings, by Griffiths–Landman–Grothendieck Theorem see [Ka70]) the operator  $T: H^m(Y_b) \rightarrow H^m(Y_b)$  is unipotent and  $(T - \text{id})^{m+1} = 0$ . It follows that the transformation (4) is also unipotent. Denote by  $N$  the logarithm of the transformation (4), which is therefore a nilpotent operator on  $H^\bullet(Y, Y_b)$ . One has  $N^{m+1} = 0$ .

**Definition 7.** We say that the Landau–Ginzburg model  $(Y, w)$  is of *Fano type* if the operator  $N$  on the relative cohomology  $H^{n+a}(Y, Y_b)$  has the following properties:

- (1)  $N^{n-|a|} \neq 0$ ,
- (2)  $N^{n-|a|+1} = 0$ .

The above definition is motivated by the expectation that the Landau–Ginzburg model of Fano type usually appears as a mirror of a projective Fano manifold  $X$  (see Subsection 2.5 below).

**Definition 8.** Assume that  $(Y, w)$  is a Landau–Ginzburg model of Fano type. Consider the relative cohomology  $H^\bullet(Y, Y_b)$  with the nilpotent operator  $N$  and the induced canonical filtration  $W$ . The Landau–Ginzburg numbers  $h^{p,q}(Y, w)$  are defined as follows:

$$\begin{aligned} h^{p,n-q}(Y, w) &= \dim gr_{2(n-p)}^{W, n-a} H^{n+p-q}(Y, Y_b) \quad \text{if } a = p - q \geq 0, \\ h^{p,n-q}(Y, w) &= \dim gr_{2(n-q)}^{W, n+a} H^{n+p-q}(Y, Y_b) \quad \text{if } a = p - q < 0. \end{aligned}$$

*Remark 9.* Our Definition 8 differs from [KKP14, Definition 3.2]

$$(5) \quad h^{p,q}(Y, w) = \dim gr_p^{W, p+q} H^{p+q}(Y, Y_b)$$

by the indices of the grading. The equation (5) seems not to be what the authors had in mind. For example according to (5)  $p$  is allowed to vary from 0 to  $2n$  and  $q$  is allowed to be negative (see Subsection 2.5 for an explanation).

**2.3. The numbers  $i^{p,q}(Y, w)$ .** Recall that for each  $\lambda \in \mathbb{A}^1$  one has the corresponding sheaf  $\phi_{w-\lambda} \mathbb{C}_Y$  of vanishing cycles for the fiber  $Y_\lambda$ . The sheaf  $\phi_{w-\lambda} \mathbb{C}_Y$  is supported on the fiber  $Y_\lambda$  and is equal to zero if  $\lambda$  is not a critical value of  $w$ . From the works of Schmid, Steenbrink, and Saito it is classically known that the constructible complex  $\phi_{w-\lambda} \mathbb{C}_Y$  carries a structure of a mixed Hodge module and so its hypercohomology inherits a mixed Hodge structure. For a mixed Hodge module  $S$  we will denote by  $i^{p,q}S$  the  $(p, q)$  Hodge numbers of the  $p + q$  weight graded piece  $gr_{p+q}^W S$ .

**Definition 10.** (1) Assume that the horizontal divisor  $D^h \subset Z$  is empty, i.e. assume that the map  $w: Y \rightarrow \mathbb{A}^1$  is proper. Then the Landau–Ginzburg Hodge numbers  $i^{p,q}(Y, w)$  are defined as follows:

$$i^{p,q}(Y, w) = \sum_{\lambda \in \mathbb{A}^1} \sum_k i^{p,q+k} \mathbb{H}^{p+q-1}(Y_\lambda, \phi_{w-\lambda} \mathbb{C}_Y).$$

(2) In the general case denote by  $j: Y \hookrightarrow Z$  the open embedding and define similarly

$$i^{p,q}(Y, w) = \sum_{\lambda \in \mathbb{A}^1} \sum_k i^{p,q+k} \mathbb{H}^{p+q-1}(Y_\lambda, \phi_{w-\lambda} \mathbf{R}j_* \mathbb{C}_Y).$$

**2.4. Conjectures.** It is proved in [KKP14] that for every  $m$  the above numbers satisfy the equalities

$$(6) \quad \dim H^m(Y, Y_b; \mathbb{C}) = \sum_{p+q=m} i^{p,q}(Y, w) = \sum_{p+q=m} h^{p,q}(Y, w) = \sum_{p+q=m} f^{p,q}(Y, w).$$

The authors state several conjectures which together refine the equalities (6). The next is a modification of Conjecture 3.6 in [KKP14], see Remark 9.

**Conjecture A.** *Assume that  $(Y, w)$  is a Landau–Ginzburg model of Fano type. Then for every  $p, q$  there are equalities*

$$h^{p,q}(Y, w) = f^{p,q}(Y, w) = i^{p,q}(Y, w).$$

The Landau–Ginzburg model  $(Y, w)$  of Fano type (together with a tame compactification) typically arises as a mirror of a projective Fano manifold  $X$ ,  $\dim X = \dim Y$ .

The following is the Conjecture 3.7 in [KKP14], see Remark 9.

**Conjecture B.** *In the above mirror situation for each  $p, q$  we have the equality*

$$f^{p,q}(Y, w) = h^{p,n-q}(X),$$

where  $h^{p,q}(X)$ ’s are the usual Hodge numbers for  $X$ .

**2.5. Explanation of Conjectures.** We refer the interested reader to [KKP14] for a detailed description of the motivation for Conjectures A and B. Basically the motivation comes from HMS, Hochschild homology identifications, and identification of the monodromy operator with the Serre functor. Namely, assume that the Landau–Ginzburg model  $(Y, w)$  as above (together with a tame compactification) is of Fano type and is a mirror of a projective Fano manifold  $X$ ,  $\dim X = \dim Y$ . Then by HMS conjecture one expects an equivalence of categories

$$(7) \quad D^b(\text{coh } X) \simeq FS(Y, w, \omega_Y),$$

where  $D^b(\text{coh } X)$  is the bounded derived category of coherent sheaves on  $X$  and  $FS(Y, w)$  is the Fukaya–Seidel category of the Landau–Ginzburg model  $(Y, w)$  with an appropriate symplectic form  $\omega_Y$ . This equivalence induces for each  $a$  an isomorphism of the Hochschild homology spaces

$$HH_a(D^b(\text{coh } X)) \simeq HH_a(FS(Y, w, \omega_Y)).$$

It is known that

$$(8) \quad HH_a(D^b(\text{coh } X)) \simeq \bigoplus_{p-q=a} H^p(X, \Omega_X^q)$$

and it is expected that

$$(9) \quad HH_a(FS(Y, w, \omega_Y)) \simeq H^{n+a}(Y, Y_b).$$

The equivalence (7) and isomorphisms (8), (9) suggest an isomorphism

$$H^{n+a}(Y, Y_b) = \bigoplus_{p-q=a} H^p(X, \Omega_X^q).$$

Moreover, the equivalence (7) identifies the Serre functors  $S_X$  and  $S_Y$  on the two categories. The functor  $S_X$  acts on the cohomology  $H^\bullet(X)$  and the logarithm of this operator is equal (up to a sign) to the cup-product with  $c_1(K_X)$ . Since  $X$  is Fano, the operator  $c_1(K_X) \cup (\cdot)$  is a Lefschetz operator on the space

$$\bigoplus_{p-q=a} H^p(X, \Omega_X^q)$$

for each  $a$ . On the other hand, the Serre functor  $S_Y$  induces an operator on the space  $H^{n+a}(Y, Y_b)$  which is the inverse of the monodromy transformation  $T$ . This suggests that the monodromy weight filtration for the nilpotent operator  $c_1(K_X) \cup (\cdot)$  on the space  $\bigoplus_{p-q=a} H^p(X, \Omega_X^q)$  should coincide with the similar filtration for the logarithm  $N$  of the operator  $T$  on  $H^{n+a}(Y, Y_b)$ . First notice that the operator  $c_1(K_X) \cup (\cdot)$  on the space  $\bigoplus_{p-q=a} H^p(X, \Omega_X^q)$  satisfies  $(c_1(K_X) \cup (\cdot))^{n-|a|} \neq 0$  by Hard Lefschetz theorem and  $(c_1(K_X) \cup (\cdot))^{n-|a|+1} = 0$ . This explains our Definition 7. Moreover, the induced filtration  $W$  on  $\bigoplus_{p-q=a} H^p(X, \Omega_X^q)$  has the properties:

$$h^{p,q}(X) = gr_{2(n-p)}^{W, n-a} \left[ \bigoplus_{p-q=a} H^p(X, \Omega_X^q) \right] \quad \text{if } a \geq 0$$

and

$$h^{p,q}(X) = gr_{2(n-q)}^{W, n+a} \left[ \bigoplus_{p-q=a} H^p(X, \Omega_X^q) \right] \quad \text{if } a < 0.$$

Thus one expects the equality of Hodge numbers

$$h^{p,n-q}(Y, w) = h^{p,q}(X),$$

which is a combination of the above conjectures.

**2.6. Summary of results.** In this work we consider tame compactified Landau–Ginzburg models  $(Z, f)$  of dimension 2. More precisely, we consider a rational elliptic surface  $f: Z \rightarrow \mathbb{P}^1$  with  $f^{-1}(\infty)$  being a reduced divisor which is a wheel of  $d$  rational curves,  $1 \leq d \leq 9$  (it is a nodal rational curve if  $d = 1$ ). In this case the horizontal divisor  $D^h$  is empty, so  $D = D^v$ . In the paper [AKO06] it is proved that the corresponding Landau–Ginzburg model  $(Y, w)$  appears as a (homological) mirror of a del Pezzo surface  $X$  of degree  $d$ . The authors also establish HMS for the case  $d = 0$ : in this case  $f^{-1}(\infty)$  is a smooth elliptic curve and  $(Y, w)$  is mirror to the blowup  $X$  of  $\mathbb{P}^2$  in 9 points of intersection of two cubic curves. Note that such  $X$  is not Fano, hence one expects that the corresponding Landau–Ginzburg model  $(Y, w)$  is not of Fano type. We confirm this prediction. The next theorem summarizes the main results of our paper.

**Theorem 11.** *Let  $f: Z \rightarrow \mathbb{P}^1$  be an elliptic surface with the reduced infinite fiber  $D = f^{-1}(\infty)$  which is a wheel of  $d$  rational curves for  $1 \leq d \leq 9$ , or is a smooth elliptic curve for  $d = 0$ . We assume that  $f$  has a section. As before put  $(Y, w) = (Z \setminus D, f|_{Z \setminus D})$ .*

- (i) *If  $1 \leq d \leq 9$ , then the Landau–Ginzburg model  $(Y, w)$  is of Fano type and there are equalities of Hodge numbers*

$$f^{p,q}(Y, w) = h^{p,q}(Y, w).$$

- (ii) *Let  $1 \leq d \leq 9$  and let  $X$  be a del Pezzo surface which is a mirror in the sense of [AKO06] to the Landau–Ginzburg model  $(Y, w)$ . There are equalities of Hodge numbers*

$$f^{p,q}(Y, w) = h^{p,2-q}(X).$$

- (iii) *If  $d = 0$ , then  $(Y, w)$  is not of Fano type.*

The proof of Theorem 11 is contained in Proposition 24, Proposition 30, and Remark 31.

Thus Conjecture A about the numbers  $f^{p,q}(Y, w)$ ,  $h^{p,q}(Y, w)$  and Conjecture B hold in case  $(Y, w)$  is of Fano type ( $1 \leq d \leq 9$ ). We will also show that in the context of Theorem 11 the numbers  $i^{p,q}(Y, w)$  are *not* equal to the numbers  $f^{p,q}(Y, w)$  (or to the numbers  $h^{p,q}(Y, w)$ , or  $h^{p,2-q}(X)$ ), therefore providing a counter example to Conjecture A, see Remark 32. We

do not know how to define the "correct" numbers  $i^{p,q}(Y, w)$ , which would make Conjecture A true.

### 3. MONODROMY ACTION ON RELATIVE COHOMOLOGY

Let  $V$  be a smooth complex algebraic variety of dimension  $n$  with a proper morphism  $w: V \rightarrow \mathbb{C}$ . Let  $b \in \mathbb{C}$  be a regular value of  $w$ . In this section we construct the monodromy action on the relative homology  $H_\bullet(V, V_b)$ , which by duality will induce the desired action on  $H^\bullet(V, V_b)$ .

Let  $S^1 \simeq C \subset \mathbb{P}^1$  be a loop passing through the point  $b$  that goes once around the  $\infty$  in the counter clockwise direction in such a way that there are no singular values of  $w$  on or inside  $C$ . Denote by  $M$  the preimage  $w^{-1}(C) \subset Y$ . Then  $M$  is a compact oriented smooth manifold which contains the fiber  $V_b$ . The (real) dimensions of  $M$  and  $V_b$  are  $2n - 1$  and  $2n - 2$  respectively. By Ehresmann's Lemma the map  $w: M \rightarrow C$  is a locally trivial fibration of smooth manifolds with the fibers diffeomorphic to  $V_b$ . Hence there exists a diffeomorphism  $T: V_b \rightarrow V_b$  such that  $M$  is diffeomorphic to the quotient

$$M = V_b \times [0, 1] / \{(a, 0) = (T(a), 1) \text{ for all } a \in V_b\}.$$

For the pair  $(M, V_b)$  we have the corresponding long exact homology sequence

$$(10) \quad \dots \rightarrow H_i(V_b) \xrightarrow{\alpha_i} H_i(M) \xrightarrow{\beta_i} H_i(M, V_b) \xrightarrow{\partial_i} H_{i-1}(V_b) \rightarrow \dots$$

The diffeomorphism  $T: V_b \rightarrow V_b$  induces automorphisms  $T: H_i(V_b) \rightarrow H_i(V_b)$ .

**Lemma 12.** *For each  $i \geq 0$ , there exists a homomorphism  $L_i: H_i(V_b) \rightarrow H_{i+1}(M, V_b)$  such that for all  $x \in H_i(V_b)$  we have*

$$\partial_{i+1} L_i(x) = T(x) - x.$$

*Proof.* Let  $z$  be an  $i$ -dimensional cycle in  $V_b$ . Consider the  $(i + 1)$ -dimensional relative cycle  $z \times [0, 1]$  in  $(V_b \times [0, 1], V_b \times \{0\} \cup V_b \times \{1\})$  with boundary  $z \times \{1\} - z \times \{0\}$ . Its image  $L_i(z)$  in  $M$  is a relative  $(i + 1)$ -cycle with boundary  $T(z) - z$  in  $V_b$ . This construction yields the required homomorphism  $L_i: H_i(V_b) \rightarrow H_{i+1}(M, V_b)$ . Given  $x \in H_i(V_b)$  the equality

$$\partial_{i+1} L_i(x) = T(x) - x$$

is clear from the construction.  $\square$

**Proposition 13.** *For each  $i \geq 0$  the map  $L_i: H_i(V_b) \rightarrow H_{i+1}(M, V_b)$  is injective.*

*Proof.* Let  $z$  be an  $i$ -cycle on  $V_b$ , which represents a nonzero homology class  $[z] \in H_i(V_b)$ . By Poincaré duality for the compact smooth manifold  $V_b$  of dimension  $2n - 2$ , there exists a  $(2n - 2 - i)$ -cycle  $z'$  on  $V_b$  such that the homological intersection  $[z'] \cdot [z]$  is non-zero. Choose a fiber  $V_\epsilon \subset M$  of  $w$  near  $V_b$ . By construction of  $M$ , the fibers  $V_b$  and  $V_\epsilon$  are canonically identified. Let  $z'_\epsilon$  be the  $(2n - 2 - i)$ -cycle on  $V_\epsilon$  corresponding to  $z'$  under this identification. We will consider  $z'_\epsilon$  as a  $(2n - 2 - i)$ -cycle in the open manifold  $M \setminus V_b$  and let  $[z'_\epsilon] \in H_{2n-2-i}(M \setminus V_b)$  be its homology class. We have the perfect Lefschetz duality pairing (see [Sp81, Theorem 6.2.19])

$$H_{2n-2-i}(M \setminus V_b) \times H_{i+1}(M, V_b) \rightarrow \mathbb{C}$$

defined by intersection of cycles. By the construction there is an equality of intersection numbers

$$[z'_\epsilon] \cdot L_i[z] = \pm [z'] \cdot [z] \neq 0.$$

Hence  $L_i[z] \neq 0$ .  $\square$

**Definition 14.** For each  $i$  define the endomorphism  $T: H_i(M, V_b) \rightarrow H_i(M, V_b)$  as  $T = \text{id} + L_{i-1}\partial_i$  and the endomorphism  $T: H_i(M) \rightarrow H_i(M)$  as  $T = \text{id}$ . (In particular  $T = \text{id}$  on  $H_0(M, V_b)$ ).

**Corollary 15.** (i) For each  $i$  the image of  $\beta_i$  is equal to the space of  $T$ -invariants  $H_i(M, V_b)^T$ .  
(ii) For each  $i$  the kernel of the map  $\alpha_i: H_i(V_b) \rightarrow H_i(M)$  contains the subspace  $(T - \text{id})H_i(V_b)$ . Hence  $\alpha_i$  factors through the space of coinvariants  $H_i(V_b)_T$ .  
(iii) The long exact sequence (10) is compatible with the endomorphisms  $T$ .

*Proof.* (i) For  $u \in H_i(M)$  we have

$$T(\beta_i(u)) = \beta_i(u) + L_{i-1}\partial_i\beta_i(u) = \beta_i(u).$$

Vice versa, let  $y \in H_i(M, V_b)^T$ . Then  $T(y) = y + L_{i-1}\partial_i(y) = y$ , i.e.  $L_{i-1}\partial_i(y) = 0$ . However  $L_{i-1}$  is injective by Proposition 13. Hence  $\partial_i(y) = 0$ , i.e.  $y$  is in the image of  $\beta_i$ .

(ii) By Lemma 12, for  $x \in H_i(V_b)$  we have

$$\partial_{i+1}L_i(x) = T(x) - x,$$

hence the image of  $\partial_{i+1}$  contains the space  $(T - \text{id})H_i(V_b)$ .

(iii) The compatibility of the maps  $\alpha_i$  and  $\beta_i$  with  $T$  follows from (i) and (ii). Let  $y \in H_{i+1}(M, V_b)$ . Then

$$\begin{aligned} \partial_{i+1}T(y) &= \partial_{i+1}(y + L_i\partial_i(y)) \\ &= \partial_{i+1}(y) + \partial_{i+1}L_i\partial_i(y) \\ &= \partial_{i+1}(y) + (T - \text{id})\partial_{i+1}(y) \\ &= T\partial_{i+1}(y). \end{aligned}$$

This proves the corollary. □

The inclusion of the pairs  $(M, V_b) \subset (V, V_b)$  induces a morphism of the homology sequences

$$\begin{array}{ccccccc} \dots & \rightarrow & H_i(M) & \rightarrow & H_i(M, V_b) & \xrightarrow{\partial_i} & H_{i-1}(V_b) \rightarrow \dots \\ & & \downarrow & & \downarrow \gamma_i & & \parallel \\ \dots & \rightarrow & H_i(V) & \rightarrow & H_i(V, V_b) & \xrightarrow{\partial_i} & H_{i-1}(V_b) \rightarrow \dots \end{array}$$

**Definition 16.** Let us define for each  $i \geq 0$  the endomorphism  $T: H_i(V, V_b) \rightarrow H_i(V, V_b)$  as the composition

$$T(y) = y + \gamma_i L_{i-1}\partial_i(y)$$

for  $y \in H_i(V, V_b)$ . In particular,  $T = \text{id}$  on  $H_0(V, V_b)$ . We also define  $T: H_i(V) \rightarrow H_i(V)$  to be the identity.

By duality this defines the operators  $T$  on the cohomology  $H^i(V_b), H^i(V, V_b), H^i(V)$ .

**Corollary 17.** The sequence

$$\dots \rightarrow H_i(V) \rightarrow H_i(V, V_b) \rightarrow H_{i-1}(V_b) \rightarrow \dots$$

is compatible with the endomorphisms  $T$ . Hence also the dual cohomology sequence

$$\dots \rightarrow H^{i-1}(V_b) \rightarrow H^i(V, V_b) \rightarrow H^i(V) \rightarrow \dots$$

is compatible with  $T$ .

*Proof.* This follows directly from the definition of the operators  $T$  together with the formula in Lemma 12. □



**Proposition 18.** (i) Assume that the morphism  $\gamma_i: H_i(M, V_b) \rightarrow H_i(V, V_b)$  is injective. Then the image of the morphism  $H_i(V) \rightarrow H_i(V, V_b)$  is the space  $H_i(V, V_b)^T$  of  $T$ -invariants.

(ii) If  $H^{2n-i-1}(V) = 0$ , then the map  $H_i(M, V_b) \rightarrow H_i(V, V_b)$  is injective. Hence by (i) the image of the morphism  $H_i(V) \rightarrow H_i(V, V_b)$  is the space  $H_i(V, V_b)^T$  of  $T$ -invariants.

*Proof.* (i) Since  $T$  acts trivially on  $H_i(V)$  and the map  $H_i(V) \rightarrow H_i(V, V_b)$  is compatible with  $T$ , its image is contained in the space  $H_i(V, V_b)^T$ . Vice versa, if  $y \in H_i(V, V_b)^T$ , then  $y = y + \gamma_i L_{i-1} \partial_i(y)$ , i.e.  $\gamma_i L_{i-1} \partial_i(y) = 0$ . Hence by our assumption  $L_{i-1} \partial_i(y) = 0$ . But  $L_{i-1}$  is injective by Proposition 13, hence  $\partial_i(y) = 0$ , i.e.  $y$  is in the image of the map  $H_i(V) \rightarrow H_i(V, V_b)$ .

(ii) Consider the commutative diagram

$$\begin{array}{ccccccccc} \dots & \rightarrow & H_i(V_b) & \rightarrow & H_i(M) & \rightarrow & H_i(M, V_b) & \xrightarrow{\partial_i} & H_{i-1}(V_b) & \rightarrow & \dots \\ & & \parallel & & \downarrow \delta_i & & \downarrow \gamma_i & & \parallel & & \\ \dots & \rightarrow & H_i(V_b) & \rightarrow & H_i(V) & \rightarrow & H_i(V, V_b) & \xrightarrow{\partial_i} & H_{i-1}(V_b) & \rightarrow & \dots \end{array}$$

By an easy diagram chasing one sees that if  $\delta_i$  is injective, then so is  $\gamma_i$ .

Let  $\Delta \subset \mathbb{P}^1$  be the closed disc containing  $\infty$  and bounded by the loop  $C$ . Let  $\Delta^0 = \Delta \setminus C$  be its interior. Finally let  $W \subset V$  be the complement of the preimage  $w^{-1}(\Delta^0)$ . Then  $W$  is a compact manifold with the boundary  $M$ . Note that the inclusion  $W \subset V$  is a homotopy equivalence since  $w$  has no singular values in  $\Delta \setminus \{\infty\}$ . Hence it suffices to prove that the map  $H_i(M) \rightarrow H_i(W)$  is injective. This map is part of the long exact sequence

$$\dots \rightarrow H_{i+1}(W, M) \rightarrow H_i(M) \rightarrow H_i(W) \rightarrow \dots$$

So it suffices to prove that if  $H^{2n-i-1}(W) = 0$ , then  $H_{i+1}(W, M) = 0$ . This follows from Lefschetz duality

$$H^{2n-q}(W) \simeq H_q(W, M)$$

for the compact oriented  $2n$ -dimensional manifold  $W$  with boundary ([Sp81, Theorem 6.2.20]).

□

#### 4. TOPOLOGY OF RATIONAL ELLIPTIC SURFACES

Now we use the notation of Section 2 for the special case which we will consider in the rest of the paper. Fix a number  $0 \leq d \leq 9$  and let  $f: Z \rightarrow \mathbb{P}^1$  be a rational elliptic surface such that  $D = D^v = f^{-1}(\infty)$  is a wheel  $I_d$  of  $d$  smooth rational curves for  $d \geq 2$ , a rational curve with one node  $I_1$  for  $d = 1$ , and a smooth elliptic curve  $I_0$  for  $d = 0$ . Assume in addition that there exists a section  $\mathbb{P}^1 \rightarrow E \subset Z$ . Recall that  $Y = Z \setminus D$ .

Since  $Z$  is rational,  $\chi(\mathcal{O}_Z) = 1$ . One has  $-K_Z = D$ , see, for instance, [ISh89, §10.2]. Hence  $c_1^2(Z) = 0$ , so by Noether's formula the topological Euler characteristic of  $Z$  is equal to 12. This means that

$$h^i(Z) = \begin{cases} 1, & i = 0, 4; \\ 10, & i = 2; \\ 0, & \text{otherwise.} \end{cases}$$

By adjunction formula  $(K_Z + E) \cdot E = 2g(E) - 2 = -2$ , so  $E^2 = -1$ .

**Lemma 19.** (i) If  $d = 0$  then

$$h^i(D) = \begin{cases} 1, & i = 0, 2; \\ 2, & i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) If  $d > 0$  then

$$h^i(D) = \begin{cases} 1, & i = 0, 1; \\ d, & i = 2; \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* The part (i) is clear. Prove the part (ii). Let  $p_1, \dots, p_d$  be the intersection points of the components of  $D$ . Let  $\pi: \tilde{D} \rightarrow D$  be the normalization. Then  $\tilde{D}$  is a disjoint union of  $d$  copies of  $\mathbb{P}^1$ . Consider an exact sequence of sheaves on  $D$

$$(11) \quad 0 \rightarrow \mathbb{C}_D \rightarrow \pi_* \pi^* \mathbb{C}_D \rightarrow \bigoplus_{i=1}^d \mathbb{C}_{p_i} \rightarrow 0,$$

where  $\mathbb{C}_{p_i}$  is a skyscraper sheaf supported at  $p_i$ . Notice that

$$\dim H^i(D, \pi_* \pi^* \mathbb{C}_D) = \dim H^i(\tilde{D}) = \begin{cases} d, & i = 0, 2; \\ 0, & i = 1. \end{cases}$$

Notice also that  $H^0(D, \mathbb{C}_D) = \mathbb{C}$  and the map  $H^0(D, \mathbb{C}_D) \rightarrow H^0(D, \pi_* \pi^* \mathbb{C}_D)$  is injective. The lemma now follows from the long exact sequence of cohomology applied to the short exact sequence (11).  $\square$

**Lemma 20.** A restriction map  $s: H^2(Z) \rightarrow H^2(D)$  is surjective.

*Proof.* Since  $Z$  is a rational surface we have  $NS(Z) \otimes \mathbb{C} = H^2(Z)$ . Also we have  $NS(D) \otimes \mathbb{C} = H^2(D)$  since  $H^2(D, \mathcal{O}_D) = 0$ . Therefore it suffices to prove that the restriction map

$$NS(Z) \otimes \mathbb{Q} \rightarrow NS(D) \otimes \mathbb{Q}$$

is surjective.

If  $d$  is 0 or 1, the curve  $D$  is irreducible, and the space  $NS(D) \otimes \mathbb{Q}$  is one-dimensional and is spanned by the first Chern class of any ample line bundle. So it suffices to take an ample line bundle on  $Z$  and restrict it to  $D$ .

So assume that  $d \geq 2$ , which means that  $D$  is a wheel of smooth rational curves. Let  $D_1, \dots, D_d$  be its irreducible components. First notice that there is an isomorphism  $NS(D) \cong \mathbb{Z}^d$  given by the map

$$\text{Pic}(D) \ni \mathcal{L} \rightarrow (\deg \mathcal{L}|_{D_1}, \dots, \deg \mathcal{L}|_{D_d}).$$

Then, for any  $i$  one has

$$-2 = \deg K_{D_i} = D_i \cdot (D_i + K_Z) = D_i^2.$$

This means that the Gram matrix  $(D_i \cdot D_j)$  is equal to

$$\begin{pmatrix} -2 & 1 & 0 & 0 & \dots & 1 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & -2 & 1 \\ 1 & \dots & 0 & 0 & 1 & -2 \end{pmatrix}.$$

To prove the asserted surjectivity it suffices to find divisors  $F_1, \dots, F_d$  on the surface  $Z$ , such that the intersection matrix  $(F_i \cdot D_j)$  is non-degenerate. The section  $E$  intersects a unique

component of  $D$ , say  $D_d$ , since  $E \cdot D = 1$ . So we have  $E \cdot D_d = 1$  and  $E \cdot D_i = 0$  for  $i \neq d$ . Taking  $F_1 = D_1, \dots, F_{d-1} = D_{d-1}, F_d = E$  one gets the intersection matrix  $(F_i \cdot D_j)$  is

$$\begin{pmatrix} -2 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & -2 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix},$$

whose determinant is equal to  $(-1)^{d-1}d$ . □

Next we compute the cohomology  $H_c^i(Y)$  of  $Y$  with compact supports.

**Lemma 21.** *The following equalities hold.*

$$h_c^i(Y) = h^i(Z, j_! \mathbb{C}_Y) = \begin{cases} 0, & i = 0, 1, 3; \\ 11 - d, & i = 2; \\ 1, & i = 4. \end{cases}$$

*Proof.* The first equality follows from the fact that  $Z$  is compact. For the second one consider the short exact sequence of sheaves

$$0 \rightarrow j_! \mathbb{C}_Y \rightarrow \mathbb{C}_Z \rightarrow \mathbb{C}_D \rightarrow 0.$$

For  $d = 0$  by Lemma 19(i) the induced long exact sequence of cohomology  $H^\bullet(Z, -)$  looks like

$$\begin{array}{ccccccc} 0 & \longrightarrow & j_! \mathbb{C}_Y & \longrightarrow & \mathbb{C}_Z & \longrightarrow & \mathbb{C}_D \longrightarrow 0 \\ H^0 & & 0 & & \mathbb{C} & \xrightarrow{r} & \mathbb{C} \\ H^1 & & ?_1 & & 0 & & \mathbb{C}^2 \\ H^2 & & ?_2 & & \mathbb{C}^{10} & \xrightarrow{s} & \mathbb{C} \\ H^3 & & ?_3 & & 0 & & 0 \\ H^4 & & \mathbb{C} & & \mathbb{C} & & 0. \end{array}$$

For  $d > 0$  by Lemma 19(ii) the same long exact sequence looks like

$$\begin{array}{ccccccc} 0 & \longrightarrow & j_! \mathbb{C}_Y & \longrightarrow & \mathbb{C}_Z & \longrightarrow & \mathbb{C}_D \longrightarrow 0 \\ H^0 & & 0 & & \mathbb{C} & \xrightarrow{r} & \mathbb{C} \\ H^1 & & ?_1 & & 0 & & \mathbb{C} \\ H^2 & & ?_2 & & \mathbb{C}^{10} & \xrightarrow{s} & \mathbb{C}^d \\ H^3 & & ?_3 & & 0 & & 0 \\ H^4 & & \mathbb{C} & & \mathbb{C} & & 0. \end{array}$$

The maps  $r$  in both cases are obviously surjective, hence  $?_1 = 0$ . By Lemma 20 the maps  $s$  are surjective. Hence  $?_2 = \mathbb{C}^{11-d}$ ,  $?_3 = 0$ . □

**Corollary 22.** *By Poincare duality for  $Y$  one has*

$$h^i(Y) = \begin{cases} 1, & \text{if } i = 0; \\ 11 - d, & \text{if } i = 2; \\ 0, & \text{if } i = 1, 3, 4. \end{cases}$$

## 5. LANDAU–GINZBURG HODGE NUMBERS FOR RATIONAL ELLIPTIC SURFACES

5.1. **The numbers  $h^{p,q}(Y, w)$ .** We keep the notation of Section 4.

Consider the long exact sequence of homology

$$\dots \rightarrow H_2(Y) \rightarrow H_2(Y, Y_b) \rightarrow H_1(Y_b) \rightarrow \dots$$

Recall that there is a compatible action of the monodromy  $T$  on each term of this sequence as explained in Section 3.

**Corollary 23.** *The image of the map  $H_2(Y) \rightarrow H_2(Y, Y_b)$  coincides with the space  $H_2(Y, Y_b)^T$  of  $T$ -invariants.*

*Proof.* In the notation of Proposition 18 we have  $n = 2$ ,  $i = 2$ , and  $H^{2n-i-1}(Y) = H^1(Y) = 0$ , see Corollary 22. Hence the assertion follows from Proposition 18(ii).  $\square$

**Proposition 24.** (i) *We have*

$$(12) \quad H^k(Y, Y_b) = \begin{cases} \mathbb{C}^{12-d}, & k = 2; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) *For  $d > 0$  the Landau–Ginzburg model  $(Y, w)$  is of Fano type and*

$$(13) \quad h^{p,q}(Y, w) = \begin{cases} 1, & (p, q) = (0, 2), (2, 0); \\ 10 - d, & (p, q) = (1, 1); \\ 0, & \text{otherwise.} \end{cases}$$

(iii) *For  $d = 0$  the Landau–Ginzburg model  $(Y, w)$  is not of Fano type.*

This proposition proves Theorem 11(iii) and computes the right hand side of the equality of Theorem 11(i).

The proof of the proposition will occupy the rest of this subsection.

**Lemma 25.** *The restriction map  $H^2(Y) \rightarrow H^2(Y_b)$  is surjective. Hence the map  $H_2(Y_b) \rightarrow H_2(Y)$  is injective.*

*Proof.* Since  $Y_b$  is a smooth projective curve,  $H^2(Y_b)$  has dimension one and is spanned by the first Chern class  $c_1(L)$  of any ample line bundle  $L$  on  $Y_b$ . It suffices to take any ample line bundle  $M$  on  $Y$ , so that its restriction  $L = M|_{Y_b}$  is also ample and  $c_1(M) \in H^2(Y)$  restricts to  $c_1(L) \in H^2(Y_b)$ .  $\square$

The equation (12) now follows from the long exact sequence of cohomology

$$\dots \rightarrow H^i(Y, Y_b) \rightarrow H^i(Y) \rightarrow H^i(Y_b) \rightarrow \dots$$

using Corollary 22, the fact that  $Y_b$  is an elliptic curve, and Lemma 25. This proves part (i) of the proposition.

To prove parts (ii) and (iii) it remains to understand the action of the monodromy  $T$  on  $H_2(Y, Y_b)$ .

Consider the part of the long exact sequence of homology

$$H_3(Y, Y_b) \rightarrow H_2(Y_b) \rightarrow H_2(Y) \rightarrow H_2(Y, Y_b) \rightarrow H_1(Y_b) \rightarrow H_1(Y).$$

We know that the map  $H_2(Y_b) \rightarrow H_2(Y)$  is injective and that  $H_1(Y) = H^1(Y)^\vee = 0$ . Hence the sequence

$$(14) \quad 0 \rightarrow H_2(Y_b) \rightarrow H_2(Y) \rightarrow H_2(Y, Y_b) \rightarrow H_1(Y_b) \rightarrow 0$$

is also exact. We have  $H_2(Y_b) = \mathbb{C}$ ,  $H_1(Y_b) = \mathbb{C}^2$ ,  $H_2(Y) = \mathbb{C}^{11-d}$ , hence the sequence (14) is isomorphic to

$$0 \rightarrow \mathbb{C} \rightarrow \mathbb{C}^{11-d} \rightarrow \mathbb{C}^{12-d} \rightarrow \mathbb{C}^2 \rightarrow 0.$$

These sequences are  $T$ -equivariant, where  $T$  acts trivially on  $H_2(Y_b)$  and  $H_2(Y)$ . By Landman's theorem  $T$  acts unipotently on  $H_1(Y_b)$ .

For  $d = 0$  the fiber  $f^{-1}(\infty)$  is smooth, hence the action of  $T$  on  $H_1(Y_b)$  is trivial. Therefore the exact sequence (14) and Corollary 23 imply that the  $T$ -action on  $H_2(Y, Y_b)$  is unipotent with two Jordan blocks of size 2 and eight blocks of size 1. This means that the Landau–Ginzburg model  $(Y, w)$  is not of Fano type, which proves (iii).

For  $d > 0$  the fiber  $f^{-1}(\infty)$  is singular, so the  $T$ -action on  $H_1(Y_b)$  is nontrivial (see [Ko63, Table 1]). Therefore the exact sequence (14) and Corollary 23 imply that the  $T$ -action on  $H_2(Y, Y_b)$  is unipotent with one Jordan block of size 3 and  $9 - d$  blocks of size 1. Therefore  $(Y, w)$  is of Fano type and equations (13) hold. This completes the proof of Proposition 24.

**5.2. The numbers  $f^{p,q}(Y, w)$ .** Recall the open embedding  $j: Y \hookrightarrow Z$ .

**Lemma 26.** *We have*

$$\Omega_Z^0(\log D) = \mathcal{O}_Z \quad \text{and} \quad \Omega_Z^2(\log D) = \mathcal{O}_Z.$$

Hence

$$\Omega_Z^0(\log D)(-D) = \Omega_Z^2(\log D)(-D) = \omega_Z.$$

*Proof.* This follows from the definition of the logarithmic complex in Subsection 2.1 and the fact that  $D$  is the anticanonical divisor.  $\square$

**Proposition 27.** *The following equalities hold.*

$$(15) \quad h^i(Z, \Omega_Z^0(\log D)(-D)) = h^i(Z, \Omega_Z^2(\log D)(-D)) = \begin{cases} 0, & i=0,1; \\ 1, & i=2, \end{cases}$$

$$(16) \quad h^i(Z, \Omega_Z^1(\log D)(-D)) = \begin{cases} 0, & i=0,2; \\ 10-d, & i=1. \end{cases}$$

*Proof.* Since the surface  $Z$  is rational one has

$$h^i(Z, \mathcal{O}_Z) = \begin{cases} 1, & i=0; \\ 0, & i=1,2. \end{cases}$$

so by Serre duality

$$h^i(Z, \omega_Z) = \begin{cases} 0, & i=0,1; \\ 1, & i=2. \end{cases}$$

So the equalities (15) follow from Lemma 26.

To prove the equality (16), notice that the complex

$$\Omega_Z^0(\log D)(-D) \rightarrow \Omega_Z^1(\log D)(-D) \rightarrow \Omega_Z^2(\log D)(-D) \rightarrow 0$$

is a resolution of the sheaf  $j_! \mathbb{C}_Y$ , see, for instance, [DI87, p. 268]. This gives the spectral sequence

$$E_1^{pq} = H^p(Z, \Omega_Z^q(\log D)(-D)),$$

which converges to  $H^{p+q}(Z, j_! \mathbb{C}_Y)$ . It is known that this spectral sequence degenerates at the term  $E_1$ , see [DI87, Corollarie 4.2.4]. This means that

$$h^\bullet(Z, j_! \mathbb{C}_Y) = h^\bullet(Z, \Omega_Z^0(\log D)(-D)) + h^{\bullet-1}(Z, \Omega_Z^1(\log D)(-D)) + h^{\bullet-2}(Z, \Omega_Z^2(\log D)(-D)).$$

Equalities (15) imply that the (numerical)  $E_1$  page of this spectral sequence is as follows:

1	$h^2$	1
0	$h^1$	0
0	$h^0$	0
$\Omega_Z^0(\log D)(-D)$	$\Omega_Z^1(\log D)(-D)$	$\Omega_Z^2(\log D)(-D)$

Therefore using Lemma 21 we obtain the equalities

$$\begin{aligned} h^0(Z, \Omega_Z^1(\log D)(-D)) &= 0, \\ h^1(Z, \Omega_Z^1(\log D)(-D)) + 1 &= 11 - d, \\ h^2(Z, \Omega_Z^1(\log D)(-D)) &= 0, \end{aligned}$$

which proves the equality (16). □

**Proposition 28.** *There are the isomorphisms*

- (i)  $\Omega_Z^0(\log D, f) = \mathcal{O}_Z(-D) = \omega_Z$ ;
- (ii)  $\Omega_Z^2(\log D, f) = \Omega_Z^2(\log D) = \mathcal{O}_Z$ .
- (iii) *There exists a short exact sequence of sheaves on  $Z$*

$$0 \rightarrow \Omega_Z^1(\log D)(-D) \rightarrow \Omega_Z^1(\log D, f) \rightarrow \mathcal{O}_D \rightarrow 0.$$

*Proof.* Let  $t$  be the local coordinate on  $\mathbb{P}^1$  at  $\infty$ . Since  $D = f^{-1}(\infty)$  has simple normal crossings, locally it is equal to the zero locus of the polynomial  $xy$  on  $\mathbb{A}^2$ . We have

$$t = \frac{1}{f(x, y)} = xy$$

and so

$$df = d\left(\frac{1}{xy}\right) = \frac{-1}{xy} \left(\frac{dx}{x} + \frac{dy}{y}\right).$$

- (i) One has  $\Omega_Z^0(\log D) = \mathcal{O}_Z$ . For any function  $g \in \mathcal{O}_Z$

$$df \wedge g = \frac{-g}{xy} \left(\frac{dx}{x} + \frac{dy}{y}\right).$$

So  $g$  should be divisible by  $xy$  to lie in  $\Omega_Z^0(\log D, f)$ .

- (ii) We have

$$\Omega_Z^2(\log D, f) = \Omega_Z^2(\log D) = \omega_Z \otimes \omega_Z^{-1} = \mathcal{O}_Z.$$

- (iii) The proof will consist of several claims.

*Claim 1.* The inclusion  $f^* \Omega_{\mathbb{P}^1}^1 \subset \Omega_Z^1$  induces an inclusion

$$f^* \Omega_{\mathbb{P}^1}^1(\log \infty) \subset \Omega_Z^1(\log D, f).$$

Indeed, the sheaf  $\Omega_{\mathbb{P}^1}^1(\log \infty)$  is locally at  $\infty$  is generated by

$$\frac{dt}{t} = \frac{d(xy)}{xy} = \frac{ydx + xdy}{xy} = \frac{dx}{x} + \frac{dy}{y},$$

which implies that  $f^*\Omega_{\mathbb{P}^1}^1(\log \infty) \subset \Omega_Z^1(\log D)$ , and then clearly

$$f^*\Omega_{\mathbb{P}^1}^1(\log \infty) \subset \Omega_Z^1(\log D, f).$$

*Claim 2.* We have the inclusion  $\Omega_Z^1(\log D)(-D) \subset \Omega_Z^1(\log D, f)$ . Indeed, a local section of  $\Omega_Z^1(\log D)(-D)$  near  $D$  is given as  $s = xy(g_1 \frac{dx}{x} + g_2 \frac{dy}{y})$ , where  $g_1, g_2 \in \mathcal{O}_Z$ . Then

$$df \wedge s = -(g_2 - g_1) \frac{dx \wedge dy}{xy}.$$

*Claim 3.* The intersection  $\Omega_Z^1(\log D)(-D) \cap f^*\Omega_{\mathbb{P}^1}^1(\log \infty)$  of subsheaves of  $\Omega_Z^1(\log D, f)$  is equal to  $f^*\Omega_{\mathbb{P}^1}^1(\log \infty)(-\infty)$ .

Indeed, the sheaf  $\Omega_{\mathbb{P}^1}^1(\log \infty)(-\infty)$  near  $\infty$  is generated by

$$t \frac{dt}{t} = xy \left( \frac{dx}{x} + \frac{dy}{y} \right) \in \Omega_Z^1(\log D)(-D) \cap f^*\Omega_{\mathbb{P}^1}^1(\log \infty).$$

Hence

$$f^*\Omega_{\mathbb{P}^1}^1(\log \infty)(-\infty) \subset \Omega_Z^1(\log D)(-D) \cap f^*\Omega_{\mathbb{P}^1}^1(\log \infty).$$

Vice versa let

$$\eta = h(t) \frac{dt}{t} \in \Omega_{\mathbb{P}^1}^1(\log \infty)$$

be such that

$$\eta = h(xy) \left( \frac{dx}{x} + \frac{dy}{y} \right) \in \Omega^1(\log D)(-D).$$

Then  $h(xy)$  is divisible by  $xy$ , i.e.  $h$  vanishes at  $\infty$  and so  $\eta \in \Omega_{\mathbb{P}^1}^1(\log \infty)(-\infty)$ .

*Claim 4.* We have the equality

$$\Omega_Z^1(\log D, f) = \Omega_Z^1(\log D)(-D) + f^*\Omega_{\mathbb{P}^1}^1(\log \infty).$$

Indeed, let  $\omega = g_1 \frac{dx}{x} + g_2 \frac{dy}{y}$ , where  $g_1, g_2 \in \mathcal{O}_Z$ , be a local section of  $\Omega_Z^1(\log D, f)$  near  $D$ . Then  $\omega = 1/2(\omega_1 + \omega_2)$ , where

$$\omega_1 = (g_1 + g_2) \frac{dx}{x} + (g_1 + g_2) \frac{dy}{y}$$

and

$$\omega_2 = (g_1 - g_2) \frac{dx}{x} + (g_2 - g_1) \frac{dy}{y}.$$

We have

$$\omega_1 = (g_1 + g_2) \frac{dt}{t} \in f^*\Omega_{\mathbb{P}^1}^1(\log \infty)$$

and hence by our assumption

$$df \wedge \omega_2 = 2(g_1 - g_2) \frac{dx \wedge dy}{(xy)^2} \in \Omega^2(\log D),$$

which implies that  $(g_1 - g_2)$  is divisible by  $xy$ , i.e.  $\omega_2 \in \Omega_Z^1(\log D)(-D)$ . This proves Claim 4.

We are ready to complete the proof of Proposition 28. First notice that the quotient  $\Omega_{\mathbb{P}^1}^1(\log \infty)/\Omega_{\mathbb{P}^1}^1(\log \infty)(-\infty)$  is isomorphic to the skyscraper sheaf  $\mathbb{C}_\infty$ . Now it follows from Claim 3 and Claim 4 that there is an isomorphism of quotients

$$\Omega_Z^1(\log D, f)/\Omega_Z^1(\log D)(-D) \simeq f^*\Omega_{\mathbb{P}^1}^1(\log \infty)/f^*\Omega_{\mathbb{P}^1}^1(\log \infty)(-\infty) = f^*\mathbb{C}_\infty = \mathcal{O}_D,$$

which proves part (iii) of the proposition.  $\square$

**Proposition 29.** *The inclusion of sheaves  $\Omega_Z^1(\log D)(-D) \subset \Omega_Z^1(\log D, f)$  induces an isomorphism of cohomology*

$$H^\bullet(Z, \Omega_Z^1(\log D)(-D)) \simeq H^\bullet(Z, \Omega_Z^1(\log D, f)).$$

*Proof.* By Proposition 30(iii) there is a short exact sequence of sheaves

$$(17) \quad 0 \rightarrow \Omega_Z^1(\log D)(-D) \rightarrow \Omega_Z^1(\log D, f) \rightarrow \mathcal{O}_D \rightarrow 0.$$

We have

$$h^i(Z, \mathcal{O}_D) = \begin{cases} 1, & i = 0, 1; \\ 0, & \text{otherwise.} \end{cases}$$

and we know by Proposition 27 that

$$h^i(Z, \Omega_Z^1(\log D)(-D)) = \begin{cases} 0, & i=0, 2; \\ 10-d, & i=1. \end{cases}$$

Hence the assertion of the proposition is equivalent to the nonvanishing of the boundary map

$$H^0(Z, \mathcal{O}_D) \rightarrow H^1(Z, \Omega_Z^1(\log D)(-D)).$$

Let  $i: E \hookrightarrow Z$  be the inclusion of a section of the elliptic surface  $Z$  and consider the restriction of the sequence (17) to  $E$ :

$$(18) \quad 0 \rightarrow i^*\Omega_Z^1(\log D)(-D) \rightarrow i^*\Omega_Z^1(\log D, f) \rightarrow i^*\mathcal{O}_D \rightarrow 0.$$

Since  $E$  is a section of the map  $f$ , it intersects  $D$  transversally (in a smooth point of  $D$ ). Therefore the sequence (18) is also short exact. We identify  $E = \mathbb{P}^1$ ; so  $E \cap D = \infty$  and  $i^*\mathcal{O}_D = \mathbb{C}_\infty$ . The map  $i^*: H^0(Z, \mathcal{O}_D) \rightarrow H^0(E, \mathbb{C}_\infty)$  is an isomorphism, therefore it suffices to prove that the boundary map

$$(19) \quad H^0(E, \mathbb{C}_\infty) \rightarrow H^1(E, i^*\Omega_Z^1(\log D)(-D))$$

is not zero.

*Claim.* The sequence (18) is isomorphic to the direct sum of short exact sequences

$$(20) \quad 0 \rightarrow \Omega_E^1(\log \infty)(-\infty) \rightarrow \Omega_E^1(\log \infty) \rightarrow \mathbb{C}_\infty \rightarrow 0$$

and

$$0 \rightarrow N_{E/Z}^*(-D) \xrightarrow{=} N_{E/Z}^*(-D) \rightarrow 0 \rightarrow 0$$

Indeed, we have the canonical short exact sequence of vector bundles on  $E$ :

$$(21) \quad 0 \rightarrow N_{E/Z}^* \rightarrow i^*\Omega_Z^1 \rightarrow \Omega_E^1 \rightarrow 0$$

with the canonical splitting  $i^*f^*: \Omega_E^1 \rightarrow \Omega_Z^1$  (we identify  $E = \mathbb{P}^1$ ).

The sequence (21) induces the following commutative diagram with exact rows

$$(22) \quad \begin{array}{ccccccc} 0 & \rightarrow & N_{E/Z}^* & \rightarrow & i^*\Omega_Z^1(\log D) & \rightarrow & \Omega_E^1(\log \infty) \rightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \rightarrow & N_{E/Z}^*(-D) & \rightarrow & i^*\Omega_Z^1(\log D, f) & \rightarrow & \Omega_E^1(\log \infty) \rightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \rightarrow & N_{E/Z}^*(-D) & \rightarrow & i^*\Omega_Z^1(\log D)(-D) & \rightarrow & \Omega_E^1(\log \infty)(-\infty) \rightarrow 0. \end{array}$$

Indeed, all rows are induced by the sequence (21) and the third one is obviously exact. The vertical maps are canonical inclusions and it is clear that the whole diagram is commutative. It remains to prove the exactness of the first two rows.

There exist local coordinates  $x, t$  on  $Z$  near  $\infty \in E$  such that

$$E = \{x = 0\} \quad \text{and} \quad D = \{t = 0\}.$$



Then  $f = \frac{1}{t}$  and the sheaf  $\Omega_Z^1(\log D)$  (resp.  $\Omega_Z^1(\log D, f)$ ) is locally generated by  $dx, \frac{dt}{t}$  (resp. by  $tdx, \frac{dt}{t}$ ). This implies the exactness of the first two rows and proves the claim.

We can now complete the two bottom rows in the diagram (22) to a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \rightarrow & 0 & \rightarrow & \mathbb{C}_\infty & \rightarrow & \mathbb{C}_\infty & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 (23) \quad 0 & \rightarrow & N_{E/Z}^*(-D) & \rightarrow & i^*\Omega_Z^1(\log D, f) & \rightarrow & \Omega_E^1(\log \infty) & \rightarrow & 0 \\
 & & \parallel & & \uparrow & & \uparrow & & \\
 0 & \rightarrow & N_{E/Z}^*(-D) & \rightarrow & i^*\Omega_Z^1(\log D)(-D) & \rightarrow & \Omega_E^1(\log \infty)(-\infty) & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 
 \end{array}$$

Now the diagram (23) considered as a short exact sequence of columns splits by the maps  $i^*f^*$  as above. This proves the claim.

Now we can complete the proof of Proposition 29. Notice that the sequence (20) is isomorphic to the natural sequence

$$0 \rightarrow \mathcal{O}_E(-2) \rightarrow \mathcal{O}_E(-1) \rightarrow \mathbb{C}_\infty \rightarrow 0,$$

where the boundary map  $H^0(E, \mathbb{C}_\infty) \rightarrow H^1(E, \mathcal{O}_E(-2))$  is an isomorphism. Hence using the above claim we find that the boundary map (19) is not zero, which completes the proof of Proposition 29.  $\square$

**Proposition 30.** *One has*

$$f^{p,q}(Y, w) = \begin{cases} 1, & (p, q) = (0, 2), (2, 0); \\ 10 - d, & (p, q) = (1, 1); \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Proposition 27 and Lemma 28 give

$$f^{p,0}(Y, w) = h^p(Z, \Omega_Z^0(\log D, f)) = h^p(Z, \omega_Z) = \begin{cases} 0, & p=0,1; \\ 1, & p=2, \end{cases}$$

$$f^{p,1}(Y, w) = h^p(Z, \Omega_Z^1(\log D, f)) = h^p(Z, \Omega_Z^1(\log D)(-D)) = \begin{cases} 0, & p=0,2; \\ 10 - d, & p=1, \end{cases}$$

and

$$f^{p,2}(Y, w) = h^p(Z, \Omega_Z^2(\log D, f)) = h^p(Z, \mathcal{O}_Z) = \begin{cases} 1, & i=0; \\ 0, & i=1,2. \end{cases}$$

$\square$

This proposition computes the left hand side of equalities from Theorem 11(i) and Theorem 11(ii), and together with Proposition 24 completes the proof of Theorem 11.

## 6. END OF PROOF OF THEOREM 11 AND DISCUSSION

In this section we keep the notation from Section 5. Studying elliptic surfaces in Section 5 is motivated by Mirror Symmetry constructions for del Pezzo surfaces from [AKO06]. The authors prove there “a half” of HMS conjecture for del Pezzo surfaces. More precise, they prove that for a general del Pezzo surface  $S_d$  of degree  $d$ ,  $1 \leq d \leq 9$ , obtained by blow up of  $\mathbb{P}^2$  in  $9 - d$  general points there exist a complexified symplectic form  $\omega_Y$  on  $(Y, w)$ , where  $(Y, w)$

has  $12 - d$  nodal singular fibers, and can  $Y$  be compactified to  $Z$  for which  $D$  is a wheel of  $d$  curves, such that

$$(24) \quad D^b(\text{coh } S_d) \simeq FS(Y, w, \omega_Y).$$

We call  $(Y, w)$  a *Landau–Ginzburg model* for  $S_d$ . We allow the case  $d = 0$  as well; in this case  $(Y, w)$  is a Landau–Ginzburg model for  $\mathbb{P}^2$  blown up in 9 intersection points of two elliptic curves, see [AKO06]. The equivalence (24) holds in this case as well.

*Remark 31.* The description of del Pezzo surface  $X$  of degree  $d$  as a blow up of  $\mathbb{P}^2$  gives the following equalities:

$$h^{p,q}(X) = \begin{cases} 1, & (p, q) = (0, 2), (2, 0); \\ 10 - d, & (p, q) = (1, 1); \\ 0, & \text{otherwise.} \end{cases}$$

This remark, together with Proposition 24 and Proposition 30, provides a proof of part (ii) of Theorem 11 and thus completes the proof of this theorem. In other words, Conjecture B and “a half” of Conjecture A hold for (mirrors of) del Pezzo surfaces.

*Remark 32.* The second part of Conjecture A does not hold already for Landau–Ginzburg model  $(Y, w)$  for  $\mathbb{P}^2$ . Indeed, one has  $h^{0,0}(Y, w) = h^{1,1}(Y, w) = h^{2,2}(Y, w) = 1$ . However the Landau–Ginzburg model  $(Y, w)$  has exactly three singular fibers, and the singular set of these fibers is a single node. Hence the numbers  $i^{p,q}(Y, w)$  are integers divisible by 3.

Another approach to Landau–Ginzburg models is given by a notion of *toric Landau–Ginzburg models*. (Basically it is related to another “arrow”  $A \rightarrow B$  of HMS complementary to the equivalence (7).) A toric Landau–Ginzburg model for a Fano variety  $X$  is a Laurent polynomial satisfying particular conditions: its periods are related to Gromov–Witten invariants for  $X$  in a specific way, it admits a Calabi–Yau compactification, and it is related to some toric degeneration for  $X$ . For precise definitions and more details see in [Prz08] and [Prz13]. Toric Landau–Ginzburg models are known for smooth toric varieties, Fano threefolds, complete intersections in projective spaces or Grassmannians (see [Gi97], [Prz13], [ILP13], [CCGGK12], [PSh14a], [PSh14b], [PSh15b]). For del Pezzo surfaces of degree greater than two toric Landau–Ginzburg model is a Laurent polynomial with support on an integral polygon having exactly one strictly internal integral point; coefficients of the polynomial are determined by a symplectic form chosen on the del Pezzo surface. There exist natural tame compactifications of these toric Landau–Ginzburg model, and one can see that Theorem 11 holds for them. In particular this means that Theorem 11 holds for a quadric surface, which is not a blow up of  $\mathbb{P}^2$ , and for Landau–Ginzburg models with singular fibers whose singularities are more complicated than just a simple node, as opposed to the case of [AKO06].

One more geometrical output of Conjecture A and Conjecture B is the following.

**Conjecture C** ([PSh15b, Conjecture 1.1], see also [GKR12]). *Let  $X$  be a Fano variety of dimension  $n$ . Let  $f_X$  be its toric Landau–Ginzburg model corresponding to an anticanonical symplectic form on  $X$ . Let  $k_{f_X}$  be a number of all components of reducible fibers (without multiplicities) of a (fiberwise) Calabi–Yau compactification for  $f_X$  minus the number of reducible fibers. One has*

$$h^{1,n-1}(X) = k_{f_X}.$$

This conjecture for del Pezzo surfaces follows from the construction of compactifications of toric Landau–Ginzburg models; it is proven for Fano threefolds of rank one (see [Prz13]) and for complete intersections (see [PSh15b]).

## REFERENCES

- [AKO06] D. Auroux, L. Katzarkov, D. Orlov, *Mirror symmetry for del Pezzo surfaces: vanishing cycles and coherent sheaves*, Invent. Math. 166 (2006), no. 3, 537–582.
- [CCGGK12] T. Coates, A. Corti, S. Galkin, V. Golyshev, A. Kasprzyk, *Mirror Symmetry and Fano Manifolds*, European Congress of Mathematics (Krakow, 2–7 July, 2012), November 2013, pp. 285–300.
- [DI87] P. Deligne, L. Illusie, *Relèvements modulo  $p^2$  et décomposition du complexe de de Rham*, Invent. Math. 89 (1987), no. 2, 247–270.
- [Gi97] A. Givental, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), 141–175, Progr. Math., 160, Birkhauser Boston, Boston, MA, 1998.
- [GKR12] M. Gross, L. Katzarkov, H. Ruddat, *Towards mirror symmetry for varieties of general type*, arXiv:1202.4042.
- [ILP13] N. Ilten, J. Lewis, V. Przyjalkowski, *Toric Degenerations of Fano Threefolds Giving Weak Landau–Ginzburg Models*, Journal of Algebra 374 (2013), 104–121.
- [ISh89] V. A. Iskovskih, I. R. Shafarevich, *Algebraic surfaces*, Algebraic geometry–2, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr., 35, VINITI, Moscow, 1989, 131–263.
- [Ka70] N. Katz, *Nilpotent connections and the monodromy theorem: applications of a result of Turrittin*, IHÉS Publ. Math., (39):175–232, 1970.
- [KKP14] L. Katzarkov, M. Kontsevich, T. Pantev, *Bogomolov–Tian–Todorov theorems for Landau–Ginzburg models*, arXiv:1409.5996.
- [Ko63] K. Kodaira, *On compact analytic surfaces. II, III*, Ann. of Math. (2) 77 (1963), 563–626; *ibid.* 78 1963 1–40.
- [Prz08] V. Przyjalkowski, *On Landau–Ginzburg models for Fano varieties*, Comm. Num. Th. Phys., Vol. 1, No. 4, 713–728 (2008).
- [Prz13] V. Przyjalkowski, *Weak Landau–Ginzburg models for smooth Fano threefolds*, Izv. Math. Vol., 77 No. 4 (2013), 135–160.
- [PSh14a] V. Przyjalkowski, C. Shramov, *Laurent phenomenon for Landau–Ginzburg models of complete intersections in Grassmannians of planes*, arXiv:1409.3729.
- [PSh14b] V. Przyjalkowski, C. Shramov, *On weak Landau–Ginzburg models for complete intersections in Grassmannians*, Russian Math. Surveys 69, No. 6, 1129–1131 (2014).
- [PSh15a] V. Przyjalkowski, C. Shramov, *On Hodge numbers of complete intersections and Landau–Ginzburg models*, Int. Math. Res. Not. IMRN, 2015:21 (2015), 11302–11332.
- [PSh15b] V. Przyjalkowski, C. Shramov, *Laurent phenomenon for Landau–Ginzburg models of complete intersections in Grassmannians*, Proc. Steklov Inst. Math., 290 (2015), 91–102.
- [Sp81] E. Spanier, *Algebraic topology*, Corrected reprint. Springer-Verlag, New York-Berlin, 1981.

Valery Lunts

Department of Mathematics, Indiana University,  
Rawles Hall, 831 East 3rd Street, Bloomington, IN 47405, USA  
vlunts@indiana.edu

Victor Przyjalkowski

Steklov Mathematical Institute of Russian Academy of Sciences,  
8 Gubkina street, Moscow 119991, Russia.  
National Research University Higher School of Economics, Russian Federation,  
AG Laboratory, HSE, 7 Vavilova str., Moscow, 117312, Russia.  
victorprz@mi.ras.ru, victorprz@gmail.com